MATH 2551 Guided Notes

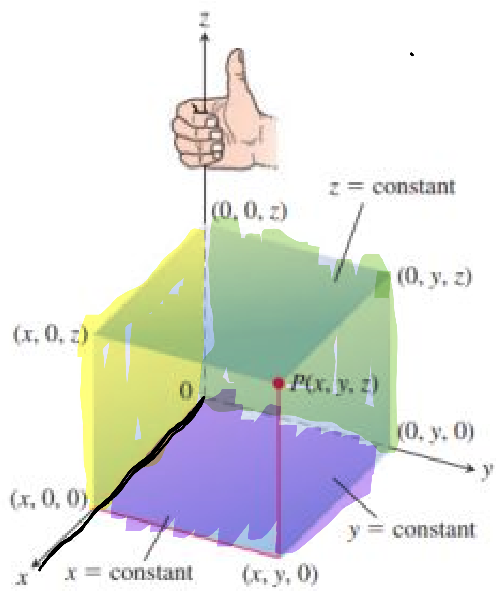
Dr. Hunter Lehmann (Dr. H, Dr. Lehmann, Dr. Hunter)

Fall 2025

Day 1 - Course Introduction and Cross Products

Pre-Lecture

12.1: Three-Dimensional Coordinates



- $\mathbb{R}^3$: (x, y, z)
- Right-handed system
- Coordinate planes
  - $x = 0$ (the yz-plane)
  - $y = 0$ (the xz-plane)
  - $z = 0$ (the xy-plane)
- Eight octants
  - 1^{st} octant has $x, y, z \geq 0$

Day 1 - Lecture

Daily Announcements & Reminders:

- Welcome to MATH 2551 G
- Use the QR code or visit pollev.com/drhgt to log into PollEverywhere
  - if you did not have an account before now, use your GT email & GTID to log in

Goals for Today:

Sections 12.1, 12.3, 12.4

- Set classroom norms
- Describe the big-picture goals of the class
- Review $\mathbb{R}^3$ and the dot product
- Introduce the cross product and its properties

Icebreaker on PollEverywhere

Introduction to the Course

Purposes:

- Pre-Lecture: Get in math headspace, first exposure to the day's topic
- Lecture: Fill in the rest of the new ideas for the week
- Studio: Guided group practice with new ideas under supervision, develop independence
- WeBWorK: Basic skills practice for the topics
- LT Practice: Extra practice problems for assessments
- Quizzes, Checkpoints, & Exams: Demonstrate learning & get feedback

Feedback loops: Lecture -> practice -> studio -> practice -> assessment -> practice -> assessment -> ...

From prereqS

- Computing deriv/integrals
  - polys, exp, log, trig
  - Deriv. Rules
  - u-sub, int. by parts
- matrix & vector operations
- linear maps as matrices

Important Canvas Items: Back to Canvas!

Big Idea: Extend differential & integral calculus.

What are some key ideas from these two courses?

Differential Calculus Integral Calculus

Poll

Before: we studied **single-variable functions** $f : \mathbb{R} \to \mathbb{R}$ like $f(x) = 2x^2 - 6$.

Now: we will study **multi-variable functions** $f : \mathbb{R}^n \to \mathbb{R}^m$: each of these functions is a rule that assigns one output vector with m entries to each input vector with n entries.

e.g. $f(x,y) = x^2 + e^y + \sin(xy)$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

$f(t) = \begin{bmatrix} 2t \\ 3t-1 \\ 4-t \end{bmatrix}$ $\quad f: \mathbb{R} \to \mathbb{R}^3$

$f(s,t) = \begin{bmatrix} s\cos(t) & s\sin(t) \\ s^2 & t^2 \end{bmatrix}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^2$

• derivatives are linear maps and so we get matrices

Example 1. What shape is the set of solutions $(x, y, z) \in \mathbb{R}^3$ to the equation

$$x^2 + y^2 = 1?$$

These solutions form a

circle radius 1] answer in $\mathbb{R}^2$

[cylinder radius 1

really really infinitely long straw

$x^2+y^2+z^2=1$

sphere of radius 1 at origin

$x^2+y^2+z=1$

paraboloid

$x^2+y^2=1$; $z=4$

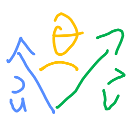
Section 12.3/4: Dot & Cross Products

Definition 2. The **dot product** of two vectors $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

This product tells us about angle between two vectors.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$



or normal or perpendicular

In particular, two vectors are **orthogonal** if and only if their dot product is 0.

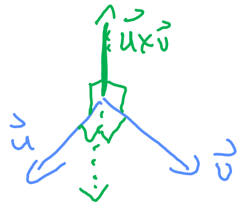
Example 3. Are $\mathbf{u} = \langle 1, 1, 4 \rangle$ and $\mathbf{v} = \langle -3, -1, 1 \rangle$ orthogonal?

$$\vec{u} \cdot \vec{v} = 1(-3) + 1(-1) + 4(1) = -3 - 1 + 4 = 0$$

so $\vec{u}$ & $\vec{v}$ are orthogonal.

Goal: Given two vectors, produce a vector orthogonal to both of them in a "nice" way.

1. Right-handed:



2. Algebraically nice:
   - $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$ & $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
   - $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

Definition 4. The **cross product** of two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$ is

$\vec{i}, \vec{j}, \vec{k}$ are standard basis vectors

$\vec{i} = \vec{e}_1 = \langle 1, 0, 0 \rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= (u_2 v_3 - v_2 u_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k}$$

Example 5. Find $\langle 1, 2, 1\rangle \times \langle 3, -1, 0\rangle$.

$$\langle 1,2,1\rangle \times \langle 3,-1,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 3 & -1 & 0 \end{vmatrix} \leftarrow \quad \det\left(\quad \right)$$

$$= \left\langle \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} \right\rangle$$

$$= \langle 0-(-1), -(0-3), -1-6\rangle$$

$$= \langle 1, 3, -7\rangle$$

$$\langle 3,-1,0\rangle \times \langle 1,2,1\rangle = -\left(\langle 1,2,1\rangle \times \langle 3,-1,0\rangle\right)$$

- antisymmetric

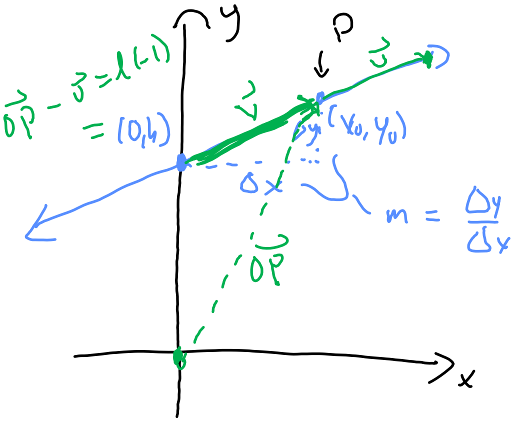
Day 2 - Lines, Planes, and Quadrics

Pre-Lecture

12.5: Lines

Lines in $\mathbb{R}^2$, a new perspective: $y = mx + b$ $\qquad y - y_0 = m(x - x_0)$

point $(0, b)$ $\qquad (x_0, y_0)$



Direction : $m = \frac{\Delta y}{\Delta x}$

In $\mathbb{R}^3$: $\frac{\Delta y}{\Delta x}$ & $\frac{\Delta z}{\Delta x}$ & $\frac{\Delta z}{\Delta y}$ & ...

A line is all points of the form

$$\ell(t) = \vec{OP} + \vec{v} \cdot t \quad \text{where}$$

$\vec{P} = (x_0, y_0, z_0)$ on the line & $\vec{v}$ is a direction for the line.

Example 7. Find a vector equation for the line that goes through the points $P = (1, 0, 2)$ and $Q = (-2, 1, 1)$.

Need:
- point on the line : $P = (1, 0, 2)$
- direction vector :

$$\vec{v} = \vec{PQ} = \langle -2-1, 1-0, 1-2 \rangle = \langle -3, 1, -1 \rangle$$

So a vector equation of the line is

$$\ell(t) = \langle -3, 1, -1 \rangle t + \langle 1, 0, 2 \rangle \qquad -\infty \le t \le \infty$$

or

$$\ell(t) = \langle -3t + 1, t, -t + 2 \rangle$$

Day 2 Lecture

Daily Announcements & Reminders:

- WW Fundamentals due F at 10 pm
- Log in to PollEverywhere with your GT email
  - password is GTID until you update it
  - after today need to login for credit
- Do warmup at pollev.com/drhgt or

Learning Targets:

- **G1: Lines and Planes.** I can describe lines using the vector equation of a line. I can describe planes using the general equation of a plane. I can find the equations of planes using a point and a normal vector. I can find the intersections of lines and planes. I can describe the relationships of lines and planes to each other. I can solve problems with lines and planes.
- **G4: Surfaces.** I can identify standard quadric surfaces including: spheres, ellipsoids, elliptic paraboloids, hyperboloids, cones, and hyperbolic paraboloids. I can match graphs of functions of two variables to their equations and contour plots and determine their domains and ranges.

Goals for Today: Sections 12.5-12.6

- Apply the cross product to solve problems
- Learn the equations that describe lines, planes, and quadric surfaces in $\mathbb{R}^3$
- Solve problems involving the equations of lines and planes
- Sketch quadric surfaces in $\mathbb{R}^3$

Example 7. Find a set of parametric equations for the line through the point $(1, 10, 100)$ which is parallel to the line with vector equation

$$\mathbf{r}(t) = \underbrace{\langle 1, 4, -3\rangle}_{\text{direction vector}} t + \underbrace{\langle 0, -1, 1\rangle}_{\text{reference point}}$$

Goal: $x(t) = 1 + t$ or $x(t) = 1 + 2t$ $t \in \mathbb{R}$

$y(t) = 10 + 4t$ $y(t) = 10 + 8t$

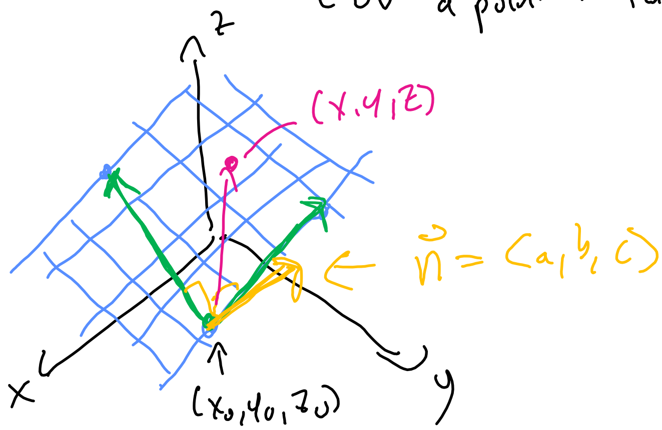
$z(t) = 100 - 3t$ $z(t) = 100 - 6t$

Section 12.5 Planes

Planes in $\mathbb{R}^3$

Conceptually: A plane is determined by either three points in $\mathbb{R}^3$ or by a single point and a direction **n**, called the *normal vector*.

[or a point & two directions



Algebraically: A plane in $\mathbb{R}^3$ has a *linear* equation (back to Linear Algebra! imposing a single restriction on a 3D space leaves a 2D linear space, i.e. a plane)

① $ax+by+cz=d$

$\vec{v} = \langle x-x_0, y-y_0, z-z_0\rangle$ is in plane so is $\perp$ to $\vec{n}$ $\Rightarrow$ $\vec{v}\cdot\vec{n}=0$

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$\vec{n} = \langle a,b,c\rangle$
(x_0,y_0,z_0) is on plane

In ①, $d = \vec{n}\cdot\langle x_0,y_0,z_0\rangle$

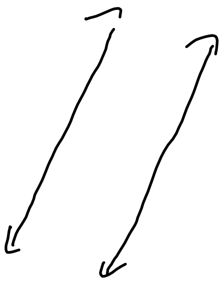
Ex: The plane w/ normal vector $\langle 1,-2,e\rangle$ containing $(\pi, \pi^2, 5)$

is $(x-\pi)-2(y-\pi^2)+e(z-5)=0$

$d = \langle 1,-2,e\rangle\cdot\langle \pi,\pi^2,5\rangle = \pi - 2\pi^2 + 5e$

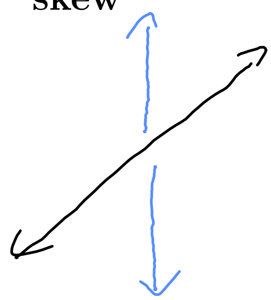
In $\mathbb{R}^3$, a pair of lines can be related in three ways:

parallel



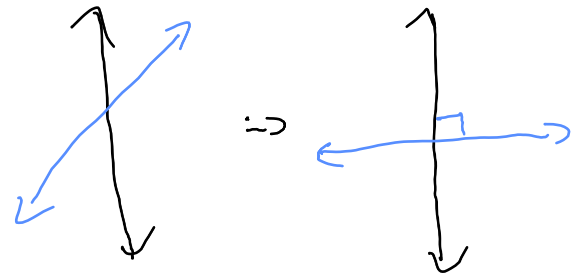
- if their direction vectors are parallel

skew



- direction vectors are not parallel & no point in common
  ↳ set equations equal (using diff. vars) & get no solution

intersecting



not orthogonal

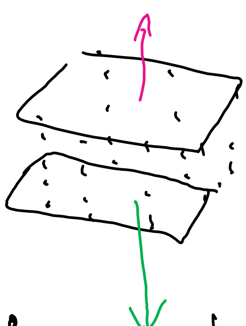
orthogonal

or get a solution

- dot prod. of dir. vectors is 0 & have point in common

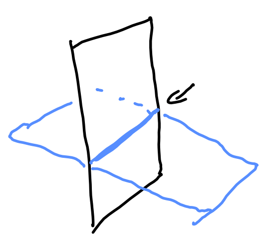
On the other hand, a pair of planes can be related in just two ways:

parallel



- normal vectors are parallel

intersecting



- sometimes orthogonal $\iff \vec{n}_1 \cdot \vec{n}_2 = 0$

- normal vectors are not parallel

Example 8. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = \langle 2, 2, 2\rangle t + \langle 0, 0, 1\rangle$$

are related in what way?

These lines are the same

because they have a point in common & are parallel

Example 9. [Poll] The lines

$$\ell_1(t) = \langle 1, 1, 1\rangle t + \langle 0, 0, 1\rangle$$

and

$$\ell_2(t) = 2t\mathbf{i} - t\mathbf{j} - t\mathbf{k}$$

are related in what way?

1) $\langle 1,1,1\rangle \cdot \langle 2,-1,-1\rangle = 2 - 1 - 1 = 0$

2) Check intersection:

$$x_1(t) \quad t = 2s \quad x_2(s)$$

$$y_1(t) \quad t = -s \quad y_2(s)$$

$$z_1(t) \quad t+1 = -s \quad z_2(s)$$

inconsistent, so skew

Example 10. Consider the planes $\overset{(1)}{y - z = -2}$ and $\overset{(2)}{x - y = 0}$. Show that the planes intersect and find an equation for the line of intersection of the planes.

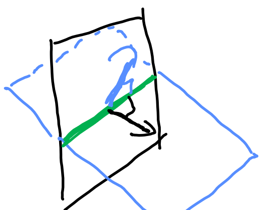
(1) Planes intersect: $\vec{n}_1 = \langle 0, 1, -1 \rangle$ & $\vec{n}_2 = \langle 1, -1, 0 \rangle$ are not parallel

OR $(0, 0, 2)$ is on both planes

(2) Find line of intersection of these planes

Need: • point (any point on both planes, e.g. $(0,0,2)$)

• direction ($\vec{n}_1 \times \vec{n}_2$)



$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \langle 0 - 1, -(0+1), (0-1) \rangle$$

$$= \langle -1, -1, -1 \rangle$$

$$\boxed{\text{line: } \ell(t) = \langle -1, -1, -1 \rangle t + \langle 0, 0, 2 \rangle, \quad t \in \mathbb{R}}$$

$y - z = 2 \rightarrow z = 2 - y$

$x - y = 0 \qquad x = y$

Section 12.6 Quadric Surfaces

On Tue.

Definition 11. A **quadric surface** in $\mathbb{R}^3$ is the set of points that solve a quadratic equation in x, y, and z.

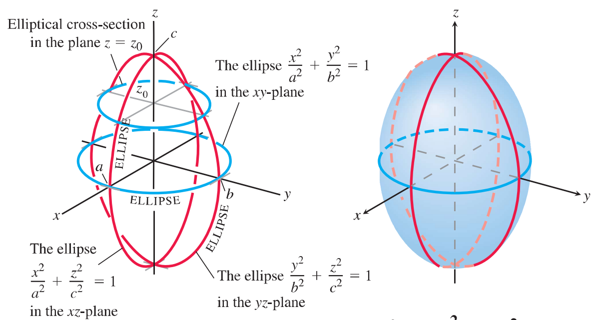
The most useful technique for recognizing and working with quadric surfaces is to examine their cross-sections. We'll also make heavy use of 3d graphing technology to get comfortable with these new objects.

Example 12. Use a 3d graphing utility to plot the quadric surface

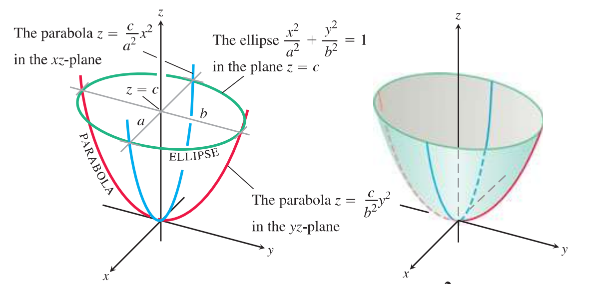
$$z = x^2 + y^2.$$

This surface is called a ____________________, because it has two coordinate directions with cross sections that are ______________ and one with cross sections that are __________.

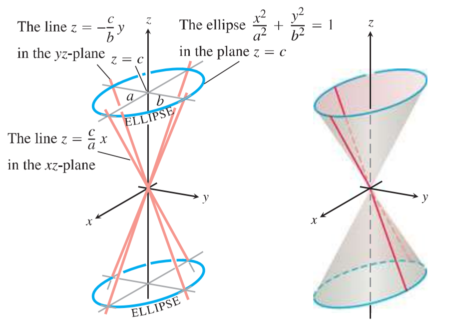
TABLE 12.1 Graphs of Quadric Surfaces



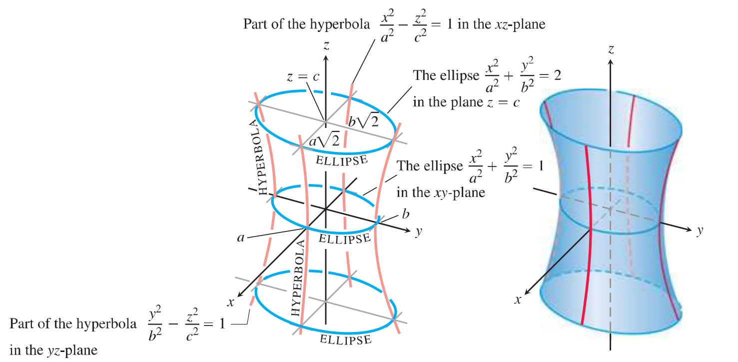
ELLIPSOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



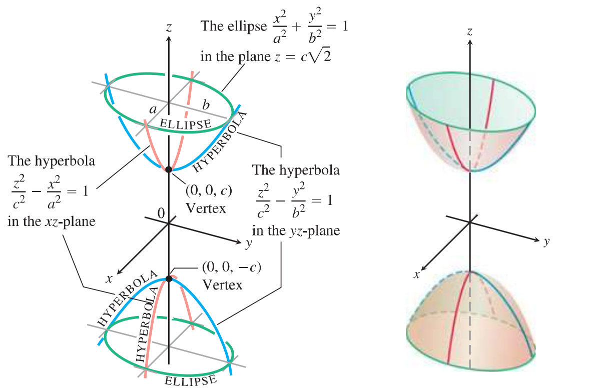
ELLIPTICAL PARABOLOID $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



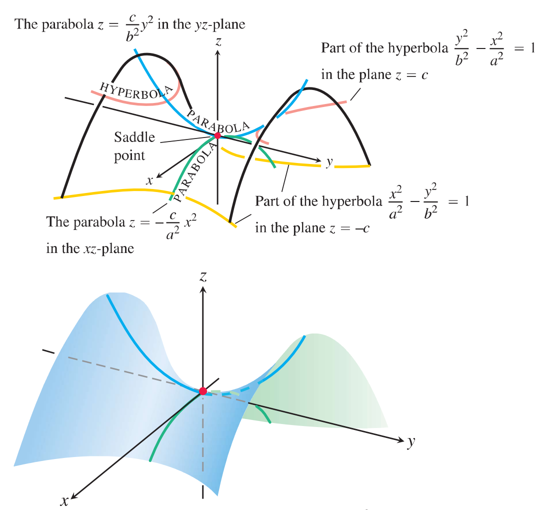
ELLIPTICAL CONE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

Example 13. [Poll] Which of the following are quadric surfaces?

1. A line
2. A sphere
3. A circle
4. An ellipse
5. The set of points (x, y, z) which solve $x^2 + y^2 - 3 = 0$.
6. The set of points (x, y, z) which solve $x^2 - y^2 - z^2 = 4$.

Example 14. [Poll] Classify the quadric surface $x + y^2 - z^2 - 3 = 0$.